

# Linear Weyl Gravity in de Sitter Universe

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## Abstract

In this paper linear Weyl gravity in de Sitter background is studied. First, linear field equation on 4-dimensional hyperboloid of de Sitter space-time, intrinsic coordinate, is obtained. In order to attain explicitly the relation between linear Weyl gravity and the unitary irreducible representation (UIR) of the de Sitter group,  $SO(1, 4)$ , we have represented field equation in ambient space notation *i.e.* five dimensional flat space notation. We have shown that linear Weyl gravity can not be associated with any UIR of the de Sitter group. We have proved that this result is obtained for the general scale-invariant equation of Boulware et al [1] as well. By using these results, we discuss that Weyl gravity can not lead to a genuine real physical theory.

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## 1 Introduction

Conformal transformations and conformal techniques have been widely used in general relativity for a long time, for example in the theory of asymptotic flatness and in the initial value formulation [2], studies of the optical geometry near black hole horizons [3] and in the other contexts [4]. It has been often claimed that conformal invariant field theories are renormalizable [5] and conformal gravity may be an alternative theory of gravity [6]. Since the gravitational fields are long range and seems to travel with light speed, in the first approximation, at least, their equations are expected to be conformally invariant. Einstein's classical theory of gravitation is not conformally invariant, thus could not be considered as a comprehensive universal

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theory of gravitational fields. The first gravitational theory, which is invariant under the scale transformation, was presented by Weyl, hence it is called Weyl gravity. The Weyl gravity leads to a theory of field with the fourth order derivative field equation. Recently, Weyl gravity has been studied in a different point of view [7].

Gravitational field, in the linear approximation, resembles a massless particle with spin 2, propagating in background space-time (background field method). Thus according to Wigner's theory, a linear gravitational field should transform with an UIR of symmetric group of space-time background. In this paper we chose de Sitter space-time background. We have shown that the linear Weyl gravity can not be associated with any UIR of de Sitter group. We have proved that this result is also obtained for the general scale-invariant equation of Boulware et al [1].

The conformal group bears a symmetry (1) under the scale invariance (dilatation), (2) under the special conformal transformations, and (3) under the Poincarè group and/or de Sitter group [8]. In previous paper we have shown that none of the UIR of the conformal group could be associated with a rank-2 symmetric tensor field  $h_{\mu\nu}$  [9]. In other words the linear Weyl gravity can not be associated to any UIR of the conformal group. As a conclusion, if a highly probable linear quantum of gravitational field exist, it could not be represented by Weyl gravity, since linear Weyl gravitational field does not transform under UIR of the conformal and de Sitter groups. Other authors have concluded the same through different methods [5, 10].

The organization of this paper is as follows. Section 2 is devoted to a brief review of the notations. The linear Weyl gravity equation in an intrinsic de Sitter (dS) coordinate is calculated in section 3. Section 4 is devoted to obtaining the field equation in ambient space notation and their relation with the UIRs of the de Sitter and conformal groups. A brief conclusion and outlook for further studies are given in section 5.

## 2 Notation

Quantum field theory in dS space has evolved as an exceedingly important subject, studied by many authors over the course of the past decade. This is due to the fact that most recent astrophysical data indicate that our universe might currently be in a dS phase. The importance of dS space has been primarily ignited by the study of the inflationary model of the universe and the quantum gravity. The de Sitter metric is a solution of the cosmological Einstein's equation with positive constant  $\Lambda$ . It is conveniently described as a hyperboloid embedded in a five-dimensional Minkowski space

$$X_H = \{x \in \mathbb{R}^5; x^2 = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2} = -\frac{3}{\Lambda}\},$$

where  $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1)$ . The dS metrics reads

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = g_{\mu\nu}^{dS} dX^\mu dX^\nu,$$

where the  $X^\mu$  are 4 space-time intrinsic coordinates of the dS hyperboloid. In this paper we take  $\alpha, \beta, \gamma, \delta, \eta = 0, 1, 2, 3, 4$  and  $\lambda, \mu, \nu, \rho = 0, 1, 2, 3$ . Any geometrical objects in this space can be written in terms of the four local coordinates  $X^\mu$  (intrinsic) or in terms of the five

global coordinates  $x^\alpha$  (ambient space). There are two Casimir operators

$$Q^{(1)} = -\frac{1}{2}L^{\alpha\beta}L_{\alpha\beta} = -\frac{1}{2}(M^{\alpha\beta} + S^{\alpha\beta})(M_{\alpha\beta} + S_{\alpha\beta}), \quad (2.1)$$

$$Q^{(2)} = -W_\alpha W^\alpha, \quad W_\alpha = -\frac{1}{8}\epsilon_{\alpha\beta\gamma\sigma\eta}L^{\beta\gamma}L^{\sigma\eta},$$

where  $M_{\alpha\beta} = -i(x_\alpha\partial_\beta - x_\beta\partial_\alpha) = -i(x_\alpha\bar{\partial}_\beta - x_\beta\bar{\partial}_\alpha)$  and the symbol  $\epsilon_{\alpha\beta\gamma\sigma\eta}$  holds for the usual antisymmetric tensor. The action of spin generator  $S_{\alpha\beta}$  is defined by [11]

$$S_{\alpha\beta}\mathcal{K}_{\gamma\delta} = -i(\eta_{\alpha\gamma}\mathcal{K}_{\beta\delta} - \eta_{\beta\gamma}\mathcal{K}_{\alpha\delta} + \eta_{\alpha\delta}\mathcal{K}_{\beta\gamma} - \eta_{\beta\delta}\mathcal{K}_{\alpha\gamma}).$$

$\bar{\partial}_\alpha$  is the tangential (or transverse) derivative in dS space,

$$\bar{\partial}_\alpha = \theta_{\alpha\beta}\partial^\beta = \partial_\alpha + H^2x_\alpha x \cdot \partial, \quad x \cdot \bar{\partial} = 0,$$

and  $\theta_{\alpha\beta}$  is the transverse projector ( $\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2x_\alpha x_\beta$ ). Following Dixmier [12], we get a classification scheme using a pair  $(p, q)$  of parameters involved in the following possible spectral values of the Casimir operators :

$$Q_p^{(1)} = (-p(p+1) - (q+1)(q-2))I_d, \quad Q_p^{(2)} = (-p(p+1)q(q-1))I_d. \quad (2.2)$$

For simplicity we define  $Q_p^{(1)} \equiv Q_p$ . Three types of scalar, tensorial or spinorial UIRs are distinguished for  $SO(1, 4)$  according to the range of values of the parameters  $q$  and  $p$  [12, 13], namely: the principal, complementary and discrete series. The flat limit indicates that the principal and complementary series value of  $p$  bears meaning of spin. For the discrete series case, the only representation which has a physically meaningful Minkowskian counterpart is  $p = q$  case. Mathematical details of the group contraction and the physical principles underlying the relationship between de Sitter and Poincaré groups can be found in Refs [14] and [15] respectively. The spin-2 tensor representations are classified with respect to the UIR of dS group as follows

- i) The UIRs  $U^{2,\nu}$  in the principal series where  $p = s = 2$  and  $q = \frac{1}{2} + i\nu$  correspond to the Casimir spectral values:

$$\langle Q_2^\nu \rangle = \nu^2 - \frac{15}{4}, \quad \nu \in \mathbb{R}, \quad (2.3)$$

note that  $U^{2,\nu}$  and  $U^{2,-\nu}$  are equivalent.

- ii) The UIRs  $V^{2,q}$  in the complementary series where  $p = s = 2$  and  $q - q^2 = \mu$ , correspond to

$$\langle Q_2^\mu \rangle = q - q^2 - 4 \equiv \mu - 4, \quad 0 < \mu < \frac{1}{4}. \quad (2.4)$$

- iii) The UIRs  $\Pi_{2,q}^\pm$  in the discrete series where  $p = s = 2$  correspond to

$$\langle Q_2^q \rangle = -6 - (q+1)(q-2), \quad q = 1, 2. \quad (2.5)$$

The “massless” spin-2 field in dS space corresponds to the  $\Pi_{2,2}^\pm$  and  $\Pi_{2,1}^\pm$  cases in which the sign  $\pm$ , stands for the helicity. In these cases, the two representations  $\Pi_{2,2}^\pm$ , in the discrete series with  $p = q = 2$ , have a Minkowskian interpretation.

The compact subgroup of conformal group  $SO(2, 4)$  is  $SO(2) \otimes SO(4)$ . Let  $C(E; j_1, j_2)$  denote the irreducible projective representation of the conformal group, where  $E$  is the eigenvalues of the conformal energy generator of  $SO(2)$  and  $(j_1, j_2)$  is the  $(2j_1 + 1)(2j_2 + 1)$  dimensional representation of  $SO(4) = SU(2) \otimes SU(2)$ . The representation  $\Pi_{2,2}^+$  has a unique extension to a direct sum of two UIRs  $C(3; 2, 0)$  and  $C(-3; 2, 0)$  of the conformal group,  $SO(2, 4)$ , with positive and negative energies respectively [14, 16]. The latter restricts to the massless Poincaré UIRs  $P^>(0, 2)$  and  $P^<(0, 2)$  with positive and negative energies respectively.  $\mathcal{P}^<(0, 2)$  (resp.  $\mathcal{P}^>(0, -2)$ ) are the massless Poincaré UIRs with positive and negative energies and positive (resp. negative) helicity. The following diagrams illustrate these connections

$$\begin{array}{ccccc} & \mathcal{C}(3, 2, 0) & & \mathcal{C}(3, 2, 0) & \hookleftarrow \mathcal{P}^>(0, 2) \\ \Pi_{2,2}^+ \hookrightarrow & \oplus & \xrightarrow{H=0} & \oplus & \oplus \\ & \mathcal{C}(-3, 2, 0) & & \mathcal{C}(-3, 2, 0) & \hookleftarrow \mathcal{P}^<(0, 2), \end{array} \quad (2.6)$$

$$\begin{array}{ccccc} & \mathcal{C}(3, 0, 2) & & \mathcal{C}(3, 0, 2) & \hookleftarrow \mathcal{P}^>(0, -2) \\ \Pi_{2,2}^- \hookrightarrow & \oplus & \xrightarrow{H=0} & \oplus & \oplus \\ & \mathcal{C}(-3, 0, 2) & & \mathcal{C}(-3, 0, 2) & \hookleftarrow \mathcal{P}^<(0, -2), \end{array} \quad (2.7)$$

where the arrows  $\hookrightarrow$  designate unique extension. It is important to note that the representations  $\Pi_{2,1}^\pm$  do not have corresponding flat limit.

### 3 Linear field equation

Consider a space-time  $(\mathcal{M}, g_{\mu\nu})$ , where  $\mathcal{M}$  is a smooth  $n$ -dimensional manifold and  $g_{\mu\nu}$  is a metric on  $\mathcal{M}$ . The following transformation

$$g_{\mu\nu} \rightarrow g'_{\mu\nu} = \Omega(x)^2 g_{\mu\nu}, \quad (3.1)$$

where  $\Omega(x)$  is a non-vanishing regular function, is called a Weyl or scale transformation. It leaves the light cones unchanged so  $(\mathcal{M}, g_{\mu\nu})$  and  $(\mathcal{M}, g'_{\mu\nu})$  have the same causal structure [2]. The general scale-invariant action in the metric signature  $(-, +, +, +)$  is [1]:

$$I_g = -\frac{1}{4} \int d^4x \sqrt{-g} (a C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} + b R^2), \quad (3.2)$$

where  $a, b$  are two constant parameters,  $g = \det(g_{\mu\nu})$  and  $C_{\mu\nu\lambda\rho}$  is the Weyl tensor, given by

$$C_{\mu\nu\lambda\rho} = R_{\mu\nu\lambda\rho} - \frac{1}{2} (g_{\mu\lambda} R_{\nu\rho} - g_{\mu\rho} R_{\nu\lambda} - g_{\nu\lambda} R_{\mu\rho} + g_{\nu\rho} R_{\mu\lambda}) + \frac{R}{6} (g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}), \quad (3.3)$$

$R_{\mu\nu\lambda\rho}$  is the Riemann tensor,  $R_{\mu\nu}$  is the Ricci tensor and  $R$  is the scalar curvature. The action can be written in the following form

$$I_g = -\frac{1}{4} \int d^4x \sqrt{-g} (2a R_{\mu\nu} R^{\mu\nu} - \frac{2a}{3} R^2 + b R^2) + (\text{surface term}). \quad (3.4)$$

Weyl gravity, which is based on the Weyl geometry action is given by ( $a = 4\alpha, b = 0$ ) [17]

$$I_w = -2\alpha \int d^4x \sqrt{-g} \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) + (\text{surface term}). \quad (3.5)$$

Therefore the total action is defined by  $I \equiv I_g + I_m$ , where  $I_m$  is a conformally invariant action of matter. Setting the variation of the total action, with respect to the metric equal to zero yields the following field equation [17]

$$2aW_{\mu\nu}^{(2)} + (b - \frac{2a}{3})W_{\mu\nu}^{(1)} = 2T_{\mu\nu}, \quad (3.6)$$

where  $T_{\mu\nu} \equiv \frac{\delta I_m}{\delta g^{\mu\nu}}$  is the energy-momentum tensor and

$$\begin{aligned} W_{\mu\nu}^{(1)} &= 2g_{\mu\nu} \nabla_\lambda \nabla^\lambda R - 2\nabla_\mu \nabla_\nu R - 2RR_{\mu\nu} + \frac{1}{2}g_{\mu\nu} R^2, \\ W_{\mu\nu}^{(2)} &= \frac{1}{2}g_{\mu\nu} \nabla_\lambda \nabla^\lambda R + \nabla_\lambda \nabla^\lambda R_{\mu\nu} - \nabla_\lambda \nabla_\nu R_\mu{}^\lambda - \nabla_\lambda \nabla_\mu R_\nu{}^\lambda - 2R_{\mu\lambda} R_\nu{}^\lambda + \frac{1}{2}g_{\mu\nu} R_{\rho\lambda} R^{\rho\lambda}. \end{aligned}$$

So the Weyl field equation can be written as follows:

$$W_{\mu\nu} = \frac{1}{4\alpha} T_{\mu\nu},$$

where

$$\begin{aligned} W_{\mu\nu} \equiv W_{\mu\nu}^{(2)} - \frac{1}{3}W_{\mu\nu}^{(1)} &= -\frac{1}{6}g_{\mu\nu} \nabla_\lambda \nabla^\lambda R + \frac{2}{3}\nabla_\mu \nabla_\nu R + \nabla_\lambda \nabla^\lambda R_{\mu\nu} - \nabla_\lambda \nabla_\nu R_\mu{}^\lambda \\ &\quad - \nabla_\lambda \nabla_\mu R_\nu{}^\lambda + \frac{2}{3}RR_{\mu\nu} - 2R_\mu{}^\lambda R_{\lambda\nu} + \frac{1}{2}g_{\mu\nu} R_{\rho\lambda} R^{\rho\lambda} - \frac{1}{6}g_{\mu\nu} R^2. \end{aligned} \quad (3.7)$$

In the background field method,  $g_{\mu\nu} = g_{\mu\nu}^{BG} + h_{\mu\nu}$ , we suppose  $g_{\mu\nu}^{BG} = g_{\mu\nu}^{dS}$ , so we have

$$g_{\mu\nu} = g_{\mu\nu}^{dS} + h_{\mu\nu}, \quad g_{\mu\nu}^{dS} \equiv \tilde{g}_{\mu\nu}, \quad g^{\mu\nu} = \tilde{g}^{\mu\nu} - h^{\mu\nu} + O(h^2). \quad (3.8)$$

The variation of Ricci tensor is [18]

$$2\delta R_{\mu\nu} = \nabla_\rho \nabla_\mu h_\nu{}^\rho + \nabla_\rho \nabla_\nu h_\mu{}^\rho - \square h_{\mu\nu} - \nabla_\mu \nabla_\nu h, \quad (3.9)$$

where  $h \equiv \tilde{g}_{\mu\nu} h^{\mu\nu}$  is the trace of  $h_{\mu\nu}$ . For the Ricci scalar, one obtains

$$R = g^{\mu\nu} R_{\mu\nu} = \tilde{R} + \delta R, \quad \delta R = \nabla_\rho \nabla_\nu h^{\nu\rho} - \square h - h^{\nu\rho} \tilde{R}_{\nu\rho}, \quad (3.10)$$

where  $\tilde{R}_{\nu\rho} = 3H^2 \tilde{g}_{\nu\rho}$  and  $\tilde{R} = 12H^2$ , (from now on for simplicity we take  $H = 1$ ).

By using the following identities,

$$\nabla_\rho \nabla_\sigma h_\nu{}^\rho = \nabla_\sigma \nabla_\rho h_\nu{}^\rho + 4h_{\sigma\nu} - g_{\nu\sigma} h, \quad (3.11)$$

$$\nabla_\lambda \square h_\rho{}^\lambda = \square \nabla_\lambda h_\rho{}^\lambda + 5\nabla_\lambda h_\rho{}^\lambda - 2\nabla_\rho h, \quad (3.12)$$

$$\nabla_\mu \nabla_\nu \square h_\lambda{}^\mu = \nabla_\nu \square \nabla_\mu h_\lambda{}^\mu + 5\nabla_\nu \nabla_\mu h_\lambda{}^\mu - 2\nabla_\nu \nabla_\lambda h + 4\square h_{\lambda\nu} - g_{\lambda\nu} \square h, \quad (3.13)$$

$W^{(1)}$  and  $W^{(2)}$  in linear approximation take the following forms:

$$W_{\mu\nu}^{(1)} = [\tilde{g}_{\mu\nu}(2\Box + 6) - 2\nabla_\mu \nabla_\nu] (\nabla_\rho \nabla_\sigma h^{\rho\sigma} - \Box h - 3h) - 24h_{\mu\nu} - 12(\nabla_\mu \nabla_\rho h_\nu^\rho + \nabla_\nu \nabla_\rho h_\mu^\rho - 2h\tilde{g}_{\mu\nu} - \Box h_{\mu\nu} - \nabla_\mu \nabla_\nu h), \quad (3.14)$$

$$W_{\mu\nu}^{(2)} = \frac{1}{2}\tilde{g}_{\mu\nu}\Box(\nabla_\rho \nabla_\sigma h^{\rho\sigma} - \Box h - 3h) + \tilde{g}_{\mu\nu}(3\nabla_\rho \nabla_\sigma h^{\rho\sigma} - 3\Box h) + \frac{1}{2}\Box(\nabla_\mu \nabla_\rho h_\nu^\rho + \nabla_\nu \nabla_\rho h_\mu^\rho + 8h_{\mu\nu} - 2h\tilde{g}_{\mu\nu} - \Box h_{\mu\nu} - \nabla_\mu \nabla_\nu h) - \nabla^\lambda \nabla_{(\nu}(\nabla_{\mu)} \nabla_\rho h_\lambda^\rho + \nabla_\lambda \nabla_\rho h_{\mu)}^\rho + 8h_{\mu\lambda} - 2h\tilde{g}_{\mu\lambda} - \Box h_{\mu\lambda} - \nabla_{\mu)} \nabla_\lambda h) - 2(3\nabla_\mu \nabla_\rho h_\nu^\rho + 3\nabla_\nu \nabla_\rho h_\mu^\rho + 15h_{\mu\nu} - 6h\tilde{g}_{\mu\nu} - 3\Box h_{\mu\nu} - 3\nabla_\mu \nabla_\nu h), \quad (3.15)$$

where we have used the usual symmetrization notation,  $2T_{(\mu\nu)} = T_{\mu\nu} + T_{\nu\mu}$ .

Therefore the linear free field equation of (3.6), turn into:

$$W_{\mu\nu}^{(2)} + \left(\frac{b}{2a} - \frac{1}{3}\right)W_{\mu\nu}^{(1)} = 0, \quad (3.16)$$

where  $W^{(1)}$  and  $W^{(2)}$  in linear approximation are defined by (3.14) and (3.15), respectively. Note that for the Weyl gravity, we impose  $b = 0$ . The linear free field equation can be rewritten in the following form:

$$D_{\mu\nu\rho\sigma}h^{\rho\sigma} = 0, \quad (3.17)$$

where

$$D_{\mu\nu\rho\sigma} = \tilde{g}_{\mu\nu}\left[-\frac{1}{6}\Box\nabla_\rho\nabla_\sigma + \frac{1}{6}\Box^2\tilde{g}_{\rho\sigma} - \frac{3}{2}\Box\tilde{g}_{\rho\sigma} + \nabla_\rho\nabla_\sigma + 18\tilde{g}_{\rho\sigma}\right] + \frac{1}{2}\Box\left[2\nabla_{(\mu}\nabla_\rho\tilde{g}_{\nu)\sigma} - \Box\tilde{g}_{\mu\rho}\tilde{g}_{\nu\sigma} - \nabla_\mu\nabla_\nu\tilde{g}_{\rho\sigma} - 2\tilde{g}_{\mu\nu}\tilde{g}_{\rho\sigma} + 20\tilde{g}_{\mu\rho}\tilde{g}_{\nu\sigma}\right] - \nabla^\lambda\nabla_{(\nu}(\nabla_{\mu)}\nabla_\rho\tilde{g}_{\lambda\sigma} + \nabla_\lambda\nabla_\rho\tilde{g}_{\mu)\sigma} - \nabla_{\mu)}\nabla_\lambda\tilde{g}_{\rho\sigma}) + \nabla_{(\nu}\Box\nabla_\rho\tilde{g}_{\mu)\sigma} + \frac{2}{3}\nabla_\mu\nabla_\nu(\nabla_\rho\nabla_\sigma - \Box\tilde{g}_{\rho\sigma} - \frac{3}{2}\tilde{g}_{\rho\sigma}) - \nabla_\nu\nabla_\mu\tilde{g}_{\rho\sigma} - 7\nabla_{(\mu}\nabla_\rho\tilde{g}_{\nu)\sigma} - 54\tilde{g}_{\mu\rho}\tilde{g}_{\nu\sigma}.$$

By imposing following conditions on  $h_{\mu\nu}$

$$h = 0, \quad \nabla_\mu h^{\mu\nu} = 0,$$

which are necessary for the physical state or the UIR of de Sitter group, we can obtain the physical part of the linear free field equation of Weyl gravity in dS space. With these conditions we get

$$W_{\mu\nu}^{(1)} = -24h_{\mu\nu} + 12\Box h_{\mu\nu},$$

$$W_{\mu\nu}^{(2)} = -\frac{1}{2}\Box^2 h_{\mu\nu} + 14\Box h_{\mu\nu} - 62h_{\mu\nu},$$

then the field equation (3.16) reduces to:

$$a\Box^2 h_{\mu\nu} - (20a + 12b)\Box h_{\mu\nu} + (24b + 108a)h_{\mu\nu} = 0, \quad (3.18)$$

and finally the linear Weyl gravity equation on the 4-dimensional de Sitter hyperboloid becomes:

$$(\Box^2 - 20\Box + 108)h_{\mu\nu} = 0. \quad (3.19)$$

## 4 Linear field equation in ambient space notation

In order to clarify the relation between field equation and the representation of the dS group, we have adopted the tensor field  $\mathcal{K}_{\alpha\beta}(x)$  in ambient space notation. In this notation, the relationship with UIRs of the dS group becomes straightforward because the Casimir operators are easily identified with the field equation [11]. The transverse tensor field  $\mathcal{K}_{\alpha\beta}(x)$  is locally determined by the “intrinsic” field  $h_{\mu\nu}(X)$  through

$$h_{\mu\nu}(X) = \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x^\beta}{\partial X^\nu} \mathcal{K}_{\alpha\beta}(x(X)). \quad (4.1)$$

The symmetric tensor field  $\mathcal{K}_{\alpha\beta}(x)$  is defined in dS space and will be viewed here as a homogeneous function, with some arbitrarily chosen degree  $\sigma$ , in the  $\mathbb{R}^5$ -variables  $x^\alpha$  [19]

$$x^\alpha \frac{\partial}{\partial x^\alpha} \mathcal{K}_{\beta\gamma}(x) = x \cdot \partial \mathcal{K}_{\beta\gamma}(x) = \sigma \mathcal{K}_{\beta\gamma}(x). \quad (4.2)$$

It also satisfies the transversality condition [19]

$$x \cdot \mathcal{K}(x) = 0, \text{ i.e. } x^\alpha \mathcal{K}_{\alpha\beta}(x) = 0, \text{ and } x^\beta \mathcal{K}_{\alpha\beta}(x) = 0. \quad (4.3)$$

To express tensor field in terms of the ambient space coordinates transverse projection is defined:

$$(Trpr\mathcal{K})_{\alpha_1 \dots \alpha_l} \equiv \theta_{\alpha_1}^{\beta_1} \dots \theta_{\alpha_l}^{\beta_l} \mathcal{K}_{\beta_1 \dots \beta_l}. \quad (4.4)$$

The transverse projection guarantees the transversality in each index. Therefore, the covariant derivative of a transverse tensor field,  $T_{\alpha_1 \dots \alpha_n}$ , in the ambient space notation becomes

$$Trpr\bar{\partial}_\beta \mathcal{K}_{\alpha_1 \dots \alpha_n} \equiv \nabla_\beta T_{\alpha_1 \dots \alpha_n} = \bar{\partial}_\beta T_{\alpha_1 \dots \alpha_n} - \sum_{i=1}^n x_{\alpha_i} T_{\alpha_1 \dots \alpha_{i-1} \beta \alpha_{i+1} \dots \alpha_n}. \quad (4.5)$$

Applying this procedure to a transverse second rank tensor field, leads to

$$\mathcal{T}_{\beta\gamma\eta} \equiv Trpr\bar{\partial}_\beta \mathcal{K}_{\gamma\eta} = \bar{\partial}_\beta \mathcal{K}_{\gamma\eta} - x_\gamma \mathcal{K}_{\beta\eta} - x_\eta \mathcal{K}_{\gamma\beta}, \quad (4.6)$$

where  $\mathcal{T}_{\beta\gamma\eta}$  is now a transverse tensor field of rank 3. For rank 3, 4 and 5 transverse tensor fields, we can respectively write:

$$\mathcal{M}_{\alpha\beta\gamma\eta} \equiv Trpr\bar{\partial}_\alpha \mathcal{T}_{\beta\gamma\eta} = \bar{\partial}_\alpha \mathcal{T}_{\beta\gamma\eta} - x_\beta \mathcal{T}_{\alpha\gamma\eta} - x_\gamma \mathcal{T}_{\beta\alpha\eta} - x_\eta \mathcal{T}_{\beta\gamma\alpha}, \quad (4.7)$$

$$\mathcal{N}_{\delta\alpha\beta\gamma\eta} \equiv Trpr\bar{\partial}_\delta \mathcal{M}_{\alpha\beta\gamma\eta} = \bar{\partial}_\delta \mathcal{M}_{\alpha\beta\gamma\eta} - x_\alpha \mathcal{M}_{\delta\beta\gamma\eta} - x_\beta \mathcal{M}_{\alpha\delta\gamma\eta} - x_\gamma \mathcal{M}_{\alpha\beta\delta\eta} - x_\eta \mathcal{M}_{\alpha\beta\gamma\delta}, \quad (4.8)$$

$$\begin{aligned} \mathcal{P}_{\epsilon\delta\alpha\beta\gamma\eta} \equiv Trpr\bar{\partial}_\epsilon \mathcal{N}_{\delta\alpha\beta\gamma\eta} = & \bar{\partial}_\epsilon \mathcal{N}_{\delta\alpha\beta\gamma\eta} - x_\delta \mathcal{N}_{\epsilon\alpha\beta\gamma\eta} - x_\alpha \mathcal{N}_{\delta\epsilon\beta\gamma\eta} - x_\beta \mathcal{N}_{\delta\alpha\epsilon\gamma\eta} - x_\gamma \mathcal{N}_{\delta\alpha\beta\epsilon\eta} \\ & - x_\eta \mathcal{N}_{\delta\alpha\beta\gamma\epsilon}. \end{aligned} \quad (4.9)$$

For example by replacing (4.6) in to (4.7), we get

$$\begin{aligned} \mathcal{M}_{\alpha\beta\gamma\eta} = Trpr\bar{\partial}_\alpha Trpr\bar{\partial}_\beta \mathcal{K}_{\gamma\eta} = & \bar{\partial}_\alpha \left( \bar{\partial}_\beta \mathcal{K}_{\gamma\eta} - 2x_{(\gamma} \mathcal{K}_{\eta)\beta} \right) - x_\beta \left( \bar{\partial}_\alpha \mathcal{K}_{\gamma\eta} - 2x_{(\gamma} \mathcal{K}_{\eta)\alpha} \right) \\ & - x_\gamma \left( \bar{\partial}_\beta \mathcal{K}_{\alpha\eta} - 2x_{(\alpha} \mathcal{K}_{\eta)\beta} \right) - x_\eta \left( \bar{\partial}_\beta \mathcal{K}_{\alpha\gamma} - 2x_{(\gamma} \mathcal{K}_{\alpha)\beta} \right). \end{aligned} \quad (4.10)$$

By using the following relations [20]

$$g_{\mu\nu}^{dS} = \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x^\beta}{\partial X^\nu} \theta_{\alpha\beta},$$

$$\nabla_\mu \cdots \nabla_\rho h_{\lambda_1 \cdots \lambda_l} = \frac{\partial x^\alpha}{\partial X^\mu} \cdots \frac{\partial x^\gamma}{\partial X^\rho} \frac{\partial x^{\eta_1}}{\partial X^{\lambda_1}} \cdots \frac{\partial x^{\eta_l}}{\partial X^{\lambda_l}} Trpr \bar{\partial}_\alpha \cdots Trpr \bar{\partial}_\gamma \mathcal{K}_{\eta_1 \cdots \eta_l},$$

we obtain

$$\nabla_\mu \nabla \cdot h_\nu = \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x^\eta}{\partial X^\nu} \mathcal{M}_{\alpha\beta\beta\eta}, \quad (4.11)$$

$$\nabla_\lambda \nabla^\lambda h_{\mu\nu} \equiv \square h_{\mu\nu} = \frac{\partial x^\gamma}{\partial X^\mu} \frac{\partial x^\eta}{\partial X^\nu} \mathcal{M}_{\alpha\alpha\gamma\eta}, \quad (4.12)$$

$$\nabla_\rho \nabla^\rho \nabla_\lambda \nabla^\lambda h_{\mu\nu} \equiv \square^2 h_{\mu\nu} = \frac{\partial x^\gamma}{\partial X^\mu} \frac{\partial x^\eta}{\partial X^\nu} \mathcal{P}_{\delta\delta\alpha\alpha\gamma\eta}, \quad (4.13)$$

where  $\mathcal{M}_{\alpha\alpha\gamma\eta}$ ,  $\mathcal{M}_{\alpha\beta\beta\eta}$  and  $\mathcal{P}_{\delta\delta\alpha\alpha\gamma\eta}$  are calculated from (4.7) and (4.9) by contraction of the indices as follows

$$\mathcal{M}_{\alpha\alpha\gamma\eta} = (\bar{\partial}^2 - 2) \mathcal{K}_{\gamma\eta} - 4x_{(\gamma} \bar{\partial} \cdot \mathcal{K}_{\eta)},$$

$$\mathcal{M}_{\alpha\beta\beta\eta} = \bar{\partial}_\eta \bar{\partial} \cdot \mathcal{K}_\gamma - \bar{\partial}_\gamma \bar{\partial} \cdot \mathcal{K}_\eta,$$

$$\mathcal{P}_{\delta\delta\alpha\alpha\gamma\eta} = \bar{\partial}^2 [(\bar{\partial}^2 - 2) \mathcal{K}_{\gamma\eta} - 4x_{(\gamma} \bar{\partial} \cdot \mathcal{K}_{\eta)}] - 4x_{(\gamma} (\bar{\partial}^2 - 6) \bar{\partial} \cdot \mathcal{K}_{\eta)} - 2(\bar{\partial}^2 - 2) \mathcal{K}_{\gamma\eta} + 8x_{(\eta} \bar{\partial} \cdot \mathcal{K}_{\gamma)}. \quad (4.14)$$

Now we are in a position to write the linear Weyl equation in ambient space notation. Using above mentioned relations, Eq.(3.16) can be written in this notation as follows

$$\begin{aligned} & \theta_{\alpha\beta} \left( \frac{1}{6} \mathcal{P}_{\delta\delta\gamma\eta\gamma\eta} + \mathcal{M}_{\gamma\eta\gamma\eta} \right) - \frac{1}{2} \left( \mathcal{P}_{\delta\delta\alpha\gamma\gamma\beta} + \mathcal{P}_{\delta\delta\beta\gamma\gamma\alpha} - \mathcal{P}_{\delta\delta\gamma\gamma\alpha\beta} - 20 \mathcal{M}_{\delta\delta\alpha\beta} \right) \\ & - \frac{2}{3} \mathcal{P}_{\alpha\beta\gamma\eta\gamma\eta} - \frac{7}{2} \left( \mathcal{M}_{\alpha\delta\delta\beta} + \mathcal{M}_{\beta\delta\delta\alpha} \right) + 54 \mathcal{K}_{\alpha\beta} \\ & + \frac{1}{2} \left( \mathcal{P}_{\delta\alpha\beta\gamma\gamma\delta} + \mathcal{P}_{\delta\beta\alpha\gamma\gamma\delta} + \mathcal{P}_{\delta\alpha\delta\gamma\gamma\beta} + \mathcal{P}_{\delta\beta\delta\gamma\gamma\alpha} - \mathcal{P}_{\alpha\gamma\gamma\delta\delta\beta} - \mathcal{P}_{\beta\gamma\gamma\delta\delta\alpha} \right) = 0, \end{aligned} \quad (4.15)$$

where  $\mathcal{K}' = 0$  and the metric signature  $(+, -, -, -)$  are imposed. Having made some calculation, we reached the following field equation

$$\begin{aligned} & -\frac{1}{2} (\bar{\partial}^4 + 16\bar{\partial}^2 + 72) \mathcal{K}_{\alpha\beta} + \bar{\partial}_{(\alpha} \bar{\partial}^2 \bar{\partial} \cdot \mathcal{K}_{\beta)} + 9\bar{\partial}_{(\alpha} \bar{\partial} \cdot \mathcal{K}_{\beta)} + 5x_{(\alpha} \bar{\partial}^2 \bar{\partial} \cdot \mathcal{K}_{\beta)} + 23x_{(\alpha} \bar{\partial} \cdot \mathcal{K}_{\beta)} \\ & - x_{(\alpha} \bar{\partial}_{\beta)} \bar{\partial} \cdot \bar{\partial} \cdot \mathcal{K} + \frac{2}{3} (\bar{\partial}_\alpha \bar{\partial}_\beta \bar{\partial} \cdot \bar{\partial} \cdot \mathcal{K} - x_\beta \bar{\partial}_\alpha \bar{\partial} \cdot \bar{\partial} \cdot \mathcal{K} + 2\bar{\partial}_\alpha \bar{\partial} \cdot \mathcal{K}_\beta - 2x_\beta \bar{\partial} \cdot \mathcal{K}_\alpha) \\ & - 4\eta_{\alpha\beta} \bar{\partial} \cdot \bar{\partial} \cdot \mathcal{K} - \frac{7}{6} \theta_{\alpha\beta} \bar{\partial} \cdot \bar{\partial} \cdot \mathcal{K} = 0. \end{aligned} \quad (4.16)$$

Useful identities in deriving this equation are:

$$x_\alpha \bar{\partial}^2 = \bar{\partial}^2 x_\alpha - 4x_\alpha - 2\bar{\partial}_\alpha,$$

$$x_\alpha \bar{\partial}^4 = \bar{\partial}^4 x_\alpha - 4\bar{\partial}^2 x_\alpha - 4(\bar{\partial}^2 - 1)\bar{\partial}_\alpha,$$

$$\bar{\partial}_\alpha \bar{\partial}^2 = \bar{\partial}^2 \bar{\partial}_\alpha + 6\bar{\partial}_\alpha - 2(\bar{\partial}^2 - 4)x_\alpha,$$



$$\bar{\partial}_\alpha \bar{\partial}_\beta = \bar{\partial}_\beta \bar{\partial}_\alpha + x_\beta \bar{\partial}_\alpha - \bar{\partial}_\beta x_\alpha + \theta_{\alpha\beta}.$$

By imposing the divergencelessness on  $\mathcal{K}$ , namely  $\bar{\partial} \cdot \mathcal{K} = 0$ , Eq.(4.16) reduces to:

$$(\bar{\partial}^4 + 16\bar{\partial}^2 + 72) \mathcal{K}_{\alpha\beta} = 0. \quad (4.17)$$

This equation can be written in terms of the second order Casimir operator of the dS group as

$$(Q_0^2 - 16Q_0 + 72) \mathcal{K}_{\alpha\beta} = 0, \quad \text{or} \quad [Q_2(Q_2 - 4) + 12] \mathcal{K}_{\alpha\beta} = 0, \quad (4.18)$$

where  $Q_0 (= -\bar{\partial}^2)$  and  $Q_2$  are the second order Casimir operators of the dS group for the scalar and ‘spin’-2 fields, respectively (Eq.(2.1)). These operators with the conditions  $x \cdot \mathcal{K} = 0$ ,  $\bar{\partial} \cdot \mathcal{K} = 0$  and  $\mathcal{K}' = 0$ , satisfy the following relation [11]

$$Q_2 \mathcal{K}_{\alpha\beta} = (Q_0 - 6) \mathcal{K}_{\alpha\beta}.$$

From the relation (2.5), it is evident that the symmetric rank-2 tensor field  $\mathcal{K}_{\alpha\beta}$ , transforms as an UIR of dS group, if it satisfies the following equations

$$(Q_2 + 4) \mathcal{K}_{\alpha\beta} = 0, \quad (Q_2 + 6) \mathcal{K}_{\alpha\beta} = 0,$$

or generally,  $\mathcal{K}_{\alpha\beta}$  satisfies

$$(Q_2 + 4)(Q_2 + 6) \mathcal{K}_{\alpha\beta} = 0. \quad (4.19)$$

Eq.(4.19) in terms of  $Q_0$  becomes:

$$Q_0(Q_0 - 2) \mathcal{K}_{\alpha\beta} = 0. \quad (4.20)$$

Clearly these equations are not compatible with equation (4.18) of the Weyl gravity. In other words linear Weyl gravity cannot be associated with any UIR of de Sitter group.

Moreover equation (4.20) can be written in the intrinsic coordinates as:

$$(\square^2 + 6\square + 8) h_{\mu\nu} = 0, \quad (4.21)$$

and in the metric signature  $(-, +, +, +)$ , we have:

$$(\square^2 - 6\square + 8) h_{\mu\nu} = 0. \quad (4.22)$$

In comparison with the Eq.(3.18), in order to associate the UIR of de Sitter group with the generally scale invariant gravitational field, we should have

$$a = -1, \quad b = \frac{7}{6}, \quad \text{and} \quad b = \frac{25}{6}.$$

These relations clearly exhibit inner contradictions and show that one can not associate any UIR of the de Sitter group with the general scale invariant gravitational field (3.18). It should be noted that in our previous work we proved a symmetric rank-2 tensor field cannot be transformed under UIR of the conformal group [9]. In other words linear Weyl gravity cannot be associated with UIRs of conformal and de Sitter groups (see (2.6) and (2.7)).

## 5 Conclusion and outlook

Conformal symmetry is indeed one of the most important measures of assessment of massless field in quantum field theory. If a graviton does exist due to its long range effect, it should have zero mass in the linear approximation. This condition immediately imposes the conformal invariance on the graviton field equations. In other words gravitational field should transform under the UIR of conformal group.

It was pointed out that Einstein's equation is not conformally invariant. In this paper it has been shown that the linear Weyl field equation does not transform according to the UIRs of de Sitter and conformal groups. Therefore Einstein's equation as well as Weyl gravitational equation are not suitable means to describe gravitational fields. In our previous work it was proved that the construction of the linear quantum gravity in de Sitter space which is invariant under conformal transformation cannot be accomplished with a rank-2 symmetric tensor field [9]. This result is in accordance with the Binetgar et al conclusion in anti-de Sitter space [21].

Barut and Böhm [8] have shown that for the physical representation of the conformal group, the eigenvalue of the conformal Casimir operator equals to 9. Binetgar et. al. [21], have proved that only the mixed symmetric rank-3 tensor field, becomes a physical representation of the conformal group. In the previous paper [9], we extended this result to de Sitter space. In other words we have shown that the conformally invariant field equation in de Sitter space should be constructed from a mixed symmetry rank-3 tensor field. We have shown that a mixed symmetry rank-3 tensor field,  $\Psi^{abc}$  with conformal degree zero, can be transformed according to the UIR of the conformal group. The linear gravitational field that is to be obtained, is simultaneously transformed under the UIR of conformal and de Sitter groups [22].

In forthcoming paper we shall present a conformally invariant gravitational field which in its linear form gives rise to the conformally invariant linear gravitational field. This may pave the road to quantization of gravitational field without any theoretical problems.

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